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On the breakdown of characteristics solutions in flows with vibrational relaxation

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The breakdown of the characteristics solution in the neighbourhood of the leading frozen characteristic is investigated for the flow induced by a piston advancing with finite acceleration into a relaxing gas and for the steady supersonic flow of a relaxing gas into a smooth compressive corner. It is found that the point of breakdown moves outwards along the leading characteristic as the relaxation time decreases and that there is no breakdown of the solution on the leading characteristic if the gas has a sufficiently small, but non-zero, relaxation time. A precise measure of this relaxation time is derived. The paper deals only with points of breakdown determined by initial derivatives of the piston path or wall shape. In the steady-flow case, the Mach number based on the frozen speed of sound must be greater than unity.

1. Introduction

In flows with a finite, non-zero relaxation time, the characteristics of the equations are those determined by the frozen flow, that is, by a flow in which the relaxation time is infinitely long. Further, the limit of the relaxation time τ tending to zero is singular. The implications are shown markedly in the following problem. A piston moves from rest with finite acceleration into a relaxing gas. We would expect the solution to break down after a critical time determined by the cusp of the envelope of the intersecting forward characteristics emanating from the piston, but that the cusp need not necessarily lie on the leading frozen characteristic. On the one hand, we can state that, if the flow is frozen, the cusp will lie on the leading frozen characteristic. On the other hand, if the gas is in equilibrium, the characteristics are no longer the frozen characteristics and the cusp will lie on the leading equilibrium characteristic. This is shown in figure 1. The velocity a_1 is the frozen speed of sound; a_2 is the equilibrium speed of sound. The development of the flow as τ decreases from infinity to zero is not immediately obvious. The present investigation is concerned with the breakdown of the solution in the neighbourhood of the leading frozen characteristic. We shall find that there exists a value of τ, strictly greater than zero, below which coincident characteristics from the origin do not intersect.

The analysis of the corresponding problem in two-dimensional steady flow is also given. In this case, a relaxing gas moving along a plane wall with a speed greater than the frozen speed of sound encounters a smooth, compressive corner.

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The solution in the neighbourhood of the first disturbed forward characteristic is discussed.

The methods used are those of Jeffrey & Tanuiti (1964). In these, the flow field is mapped onto itself, points at which characteristics of the same family intersect appearing as critical points of the mapping. We shall describe the methods in the particular context of the problem under consideration, for the reader's convenience; this is not intended to be either a rigorous development or a full exposition. For a more detailed explanation, the reader should consult Jeffrey & Tanuiti (1964); those readers who wish to savour the full subtlety of the techniques must consult Courant & Hilbert (1962).

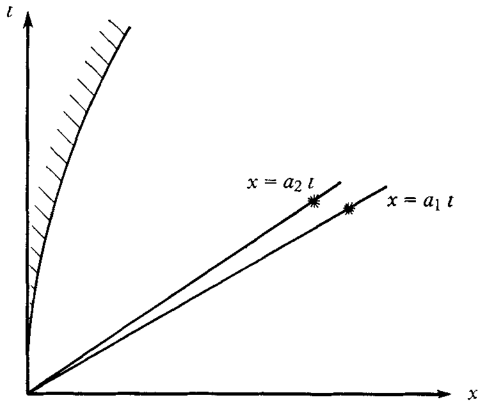

FIGURE 1

2. One-dimensional unsteady flow

In formulating the equations of motion, we use the heat-sink analogy of Johannesen (1961) which exploits the exact correspondence between the flow of a gas of variable specific heats with relaxation and the flow of a gas of constant specific heats, Johannesen's alpha gas, to which heat is added or from which heat is extracted at the rate at which energy is released from or absorbed by the vibrational mode of the real gas. Φ is the relaxation frequency and the symbols p, u, ρ, S, T and a are the properties of the alpha gas with the meanings usually attributed to them in gas-dynamics. It should be noted that T is the translational temperature and that the alpha gas is involved in a non-isentropic process even in the absence of shock waves. The quantities p, u, ρ and T, the translational properties of the gas, are unambiguously defined; the sound speed a is identical to the frozen sound speed of the real gas, but the entropy S is a property only of the fictitious alpha gas; it is related to p and ρ by the equation

$$p = \rho^{\gamma} \exp\{(S - S')/C_v\}.$$

It does not, in general, bear any simple relation to the properties of the real gas. The quantity σ is defined by the fourth equation in terms of the translational properties of the gas. The technique involves precisely the approximation inherent in considering a perfect gas with shock waves to be in thermodynamic equilibrium, and no more.

The equations are

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x} = 0,$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x} = -\frac{1}{T}\frac{D\sigma}{Dt},$$

$$\frac{\partial \sigma}{\partial t}+u\frac{\partial \sigma}{\partial x} = \rho\Phi(T)\,(\overline{\sigma}-\sigma),$$

where σ denotes the vibrational energy and $\overline{\sigma}$ its local equilibrium value.

Recasting the equations, they take the form

$$U_t+AU_x+B = 0,$$

where $$U = \{a\,u\,S\,\sigma\},$$

$$B = \rho\Phi(\overline{\sigma}-\sigma)\left\{\frac{a}{2C_v T}\;\; 0\;\; \frac{1}{T}\;\; -1\right\},$$

and

$$A = \begin{pmatrix} u & \tfrac{1}{2}(\gamma-1)\,a & 0 & 0 \\ \dfrac{2a}{\gamma-1} & u & \dfrac{a^2}{-\gamma(\gamma-1)\,C_v} & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix},$$

where we use braces { } to denote a column vector. Let λ^i be the eigenvalues of A; then $\lambda^{1,2} = u\pm a$, $\lambda^{3,4} = u$. Let L^i be the corresponding left eigenvectors; then

$$L^{1,2} = \left(\pm\frac{2}{\gamma-1}\;\; 1\;\; \mp\frac{a}{\gamma(\gamma-1)\,C_v}\;\; 0\right),$$

$$L^3 = (0\;0\;1\;0),$$

and $$L^4 = (0\;0\;0\;1).$$

Let $\phi(x,t) = 0$ be a wave front, with $\phi > 0$ corresponding to the region ahead of the wave. Let us introduce co-ordinates (t',ϕ) where $t' = t$ and $\phi_t+\lambda^\phi\phi_x = 0$, where λ^ϕ is the eigenvalue corresponding to the wave front. Then

$$L^j(U_t+\lambda^j U_x)+L^j B = 0,$$

where we have used the relation $L^j A = \lambda^j L^j$. That is, $L^j dU+L^j B\,dt = 0$ along $dx/dt = \lambda^j$, the generalization of the Riemann invariants. Transforming from (t,x) to (t',ϕ) co-ordinates, this becomes

$$L^j\left(x_\phi\frac{\partial}{\partial t'}+(\lambda^j-\lambda^\phi)\frac{\partial}{\partial \phi}\right)U+L^j B x_\phi = 0. \tag{1}$$

If $\lambda^j = \lambda^\phi$, then we have

$$L^\phi U_{t'}+L^\phi B = 0.$$

We concern ourselves here with waves for which λ^ϕ is a single root; the general case is unnecessarily comprehensive for our purposes.

We may use this relation to establish a feeling for the physics of the phenomenon by investigating the infinitesimal triangle near $t = 0$ bounded by the frozen forward characteristic from $t = 0$, the piston path, which is a particle path, and a suitable backward characteristic (see figure 2). One can show after a little manipulation that the presence of relaxation reduces the magnitude of $u + a$ at B compared with its value in the absence of relaxation. That is, the point of intersection of the positive characteristics from O and B moves outwards.

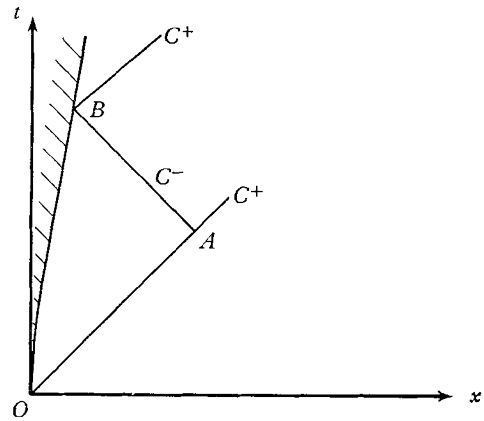

FIGURE 2

Returning to the analysis; the solution may have discontinuities with respect to ϕ at $\phi = 0$, so that U and $U_{t'}$ are both continuous at $\phi = 0$, but there may be a jump $\Pi(t') = [U_\phi]_{\phi=0+}^{\phi=0-}$ in U_ϕ and a jump $X(t') = [x_\phi]_{\phi=0+}^{\phi=0-}$ in x_ϕ. The essential point of the analysis is contained in the equation

$$X + x_\phi|_{\phi=0+} = x_\phi|_{\phi=0-}.$$

For, $x_\phi|_{\phi=0+}$, which we denote by $(x_\phi)_0$, is determined by conditions ahead of the wave and is finite. Therefore, $x_\phi|_{\phi=0-}$ is non-zero, that is the transformation is non-singular, if and only if $X + (x_\phi)_0 \neq 0$ and is finite. We see that, since $x_\phi = 1/\phi_x$, then along the curve $\phi = 0$ we have $U_x = U_\phi/x_\phi$, so that if $x_\phi \to 0$ for finite U_ϕ, U is no longer continuous.

If U_0 corresponds to conditions ahead of the wave, then $B(U_0) = 0$. From (1) with $\lambda^j \neq \lambda^\phi$, differencing across $\phi = 0$ gives

$$L_0^j \Pi = 0 \quad (\lambda^j \neq \lambda^\phi), \tag{2}$$

and from (1) with $\lambda^j = \lambda^\phi$, we have

$$L^\phi U_{t'} + L^\phi B = 0.$$

Differentiating with respect to ϕ, we have

$$L^\phi (U_\phi)_{t'} + [(\nabla_u L^\phi)\, U_\phi]'\, U_{t'} + [\nabla_u (L^\phi B)]\, U_\phi = 0,$$

where a dash superscript denotes the transpose, and ∇_u denotes the gradient operator in U-space. Across the wave front, this gives

$$L_0^\phi \Pi_{t'} + [\nabla_u (L^\phi B)]\, \Pi = 0, \tag{3}$$

since $U_{t'}$ is continuous and equal to zero on $\phi = 0+$. Now, along ϕ = constant, we have that $\partial x/\partial t' = \lambda^\phi$, so that

$$\frac{\partial}{\partial t'}(x_\phi) = \frac{\partial}{\partial \phi}\left(\frac{\partial x}{\partial t'}\right) = (\nabla_u \lambda^\phi)\, U_\phi.$$

Differencing across $\phi = 0$, we obtain

$$X_{t'} = (\nabla_u \lambda^\phi)_0\, \Pi.$$

Let us now define the limit $\tilde{Q}$ of any quantity Q by the relation $\tilde{Q} = \lim_{t'\to 0} Q$ where the limiting process is carried out in $\phi = 0-$ for all continuous quantities and on $\phi = 0$ for all jump quantities. Then, integrating the above relation, we have

$$X = \tilde{X} + \int_0^t (\nabla_u \lambda^\phi)_0\, \Pi\, dt'.$$

Now $\tilde{X} = \tilde{x}_\phi - (\tilde{x}_\phi)_0$, so that

$$X + (x_\phi)_0 = \tilde{x}_\phi + \int_0^t (\nabla_u \lambda^\phi)_0\, \Pi\, dt'.$$

Therefore, if we define t_c by the relation

$$0 = \tilde{x}_\phi + \int_0^{t_c} (\nabla_u \lambda^\phi)_0\, \Pi\, dt',$$

we see that $X + (x_\phi)_0$ is zero, that is $x_\phi|_{\phi=0-}$ is zero, at $t = t_c$. Hence, $t = t_c$ is a critical point of the mapping, that is, at $t = t_c$ the transformation has become singular, so that the characteristics in the neighbourhood of $\phi = 0$ have intersected.

Since $\tilde{\Pi} = \tilde{U}_x \tilde{x}_\phi$, the relation for t_c may be written in the form

$$0 = \tilde{\Pi}/\tilde{U}_x + \int_0^{t_c} (\nabla_u \lambda^\phi)\, \Pi\, dt'.$$

(2) gives
$$\Pi_3 = \Pi_4 = 0,$$
and
$$-\frac{2}{\gamma-1}\Pi_1 + \Pi_2 = 0,$$
and since
$$L^1 B = \rho\Phi(\bar{\sigma}-\sigma)\, a/(\gamma C_v T),$$
we have that
$$[\nabla_u(L^1 B)]_0 = \left(\frac{\rho\Phi a}{\gamma C_v T}\right)_0 \left\{\left.\frac{\partial\bar{\sigma}}{\partial a}\right|_0 \;\; 0 \;\; 0 \;\; -1\right\},$$

where we have used the fact that, in the undisturbed region ahead of the wave, $\sigma = \bar{\sigma}$. From (3), we have

$$\frac{2}{\gamma-1}(\Pi_1)_{t'} + (\Pi_2)_{t'} + \left(\frac{\rho\Phi a}{\gamma C_v T}\frac{\partial\bar{\sigma}}{\partial a}\right)_0 \Pi_1 = 0.$$

Hence
$$\Pi = \tilde{\Pi}_2 \exp(-c^* t')\{\tfrac{1}{2}(\gamma-1) \;\; 1 \;\; 0 \;\; 0\}, \tag{4}$$
where
$$c^* = \frac{\gamma-1}{4\gamma}\left(\frac{\rho\Phi a}{C_v T}\frac{\partial\bar{\sigma}}{\partial a}\right)_0,$$
which is greater than zero. Also,

$$(\nabla_u \lambda^1)_0\, \Pi = \tfrac{1}{2}(\gamma+1)\, \tilde{\Pi}_2 \exp(-c^* t'),$$

so that the equation for t_c becomes

$$1+\tfrac{1}{2}(\gamma+1)\int_0^{t_c}\tilde{u}_x \exp(-c^*t')\,dt' = 0,$$

or

$$t_c = -\frac{1}{c^*}\log\left(1-\frac{2}{\gamma+1}(c^*/-\tilde{u}_x)\right).$$

There are two interesting limiting cases. First, $c^* = 0$, corresponding to $\Phi = 0$, or completely frozen flow. Then $t_c = [2/(\gamma+1)](1/-\tilde{u}_x)$, which is positive for an advancing piston and negative for a receding piston. Now

$$\frac{\partial u}{\partial t}+\frac{2}{\gamma-1}a\frac{\partial a}{\partial x}+u\frac{\partial u}{\partial x}-\frac{a^2}{\gamma(\gamma-1)C_v}\frac{\partial S}{\partial x} = 0,$$

so that

$$\tilde{u}_t+\frac{2}{\gamma-1}a_0\tilde{a}_x = 0,$$

since $u_0 = 0$ and $(S_x)_0 = 0$. We have already shown that

$$\frac{2}{\gamma-1}\Pi_1 = \Pi_2,$$

so that

$$\frac{2}{\gamma-1}\tilde{a}_x = \tilde{u}_x.$$

Hence $\tilde{u}_t = -a_0\tilde{u}_x$. Substituting for $\tilde{u}_x$, we have that $t_c = [2/(\gamma+1)](a_0/\tilde{u}_t)$, which is the familiar result for the position of the cusp in terms of the initial piston acceleration. For flows with large relaxation times, so that $c^*/(-\tilde{u}_x) \ll 1$, we may expand the logarithm in powers of $\epsilon = c^*/(-\tilde{u}_x)$ to obtain

$$t_c = (t_c)_\infty[1+2\epsilon/(\gamma+1)+\ldots],$$

where $(t_c)_\infty = [2/(\gamma+1)(1/-\tilde{u}_x)$ is the value of t_c in a completely frozen flow. Later, we shall use $(t_c)_\infty$ as a yardstick by which the magnitude of the relaxation time may be measured. For a given piston path, $(t_c)_\infty$ is a constant.

Secondly, we note that the argument of the logarithm, $1-[2/(\gamma+1)](c^*/-\tilde{u}_x)$, is less than or equal to unity and decreases as c^* increases for a given piston path. When $c^* = \tfrac{1}{2}(\gamma+1)(-\tilde{u}_x)$, the argument is zero, that is t_c is infinite, so that there is no breakdown in the neighbourhood of $\phi = 0$. This relation is a relation between the thermodynamic properties of the gas and the initial acceleration of the piston. In particular, it implies that, for a given piston path, the critical time t_c increases from its minimum value $(t_c)_\infty$ as τ decreases from infinity and becomes infinitely large at a finite non-zero value of τ, say τ_m.

We are now in a position to view in its entirety the effect of vibrational relaxation on simple waves. For a rarefaction wave in a perfect gas, gradients of flow properties decrease along forward characteristics like $1/t$, by virtue of the divergence of the set of straight characteristics. In the presence of relaxation, Wood & Parker (1958), among others, have shown that the gradients of flow properties decay exponentially; the more quickly relaxing the gas, the more rapid the decay. In the compressive case, for piston paths with finite initial accelerations and suitably chosen higher initial derivatives, gradients of flow properties in a perfect gas increase along the leading characteristic like $1/(t_c - t)$ becoming infinite at the cusp. In the presence of relaxation, the effect of the convergence of the characteristics is modified, as (4) shows, by a simultaneous exponential decay of derivatives, the more quickly relaxing the gas, the more rapid the decay except in the neighbourhood of the point of intersection of the characteristics; this point may or may not exist.

To extend the analysis to investigate the interior of the flow field requires a knowledge of the flow field behind the initial wave. The method has been applied by Jeffrey (1964) to waves propagating into a non-uniform medium, but one whose non-uniformity is prescribed. The situation here is much more complicated.

3. Two-dimensional steady flow

We now outline the analysis for two-dimensional steady flow into a smooth compressive corner. The equations are

$$\frac{1}{q}\frac{\partial q}{\partial s} + \frac{1}{\rho}\frac{\partial \rho}{\partial s} + \frac{\partial \theta}{\partial n} = 0,$$

$$q\frac{\partial q}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} = 0,$$

$$q^2\frac{\partial \theta}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial n} = 0,$$

$$q\frac{\partial S}{\partial s} = -\rho\Phi(\bar{\sigma} - \sigma)/T,$$

$$q\frac{\partial \sigma}{\partial s} = \rho\Phi(\bar{\sigma} - \sigma),$$

where s measures distance along and n distance normal to a streamline, and θ is the deflexion of the streamline from a suitable reference direction; q is the velocity. Recasting the equations in terms of q, a, θ, S and σ, we have

$$U_s + AU_n + B = 0,$$

where $$U = \{q\ a\ \theta\ S\ \sigma\},$$

$$B = \frac{\rho\Phi(\bar{\sigma} - \sigma)}{M^2 - 1}\left\{\frac{-1}{\gamma C_v T}\ \frac{\gamma M^2 - 1}{2\gamma C_v MT}\ 0\ \frac{M^2 - 1}{aMT}\ \frac{1 - M^2}{aM}\right\},$$

and

$$A = \begin{pmatrix} 0 & 0 & \dfrac{-q}{M^2 - 1} & 0 & 0 \\ 0 & 0 & \dfrac{\gamma - 1}{2}a\dfrac{M^2}{M^2 - 1} & 0 & 0 \\ 0 & \dfrac{2}{(\gamma - 1)aM^2} & 0 & \dfrac{-1}{\gamma(\gamma - 1)C_v M^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of A are $\lambda^{1,2} = \pm(M^2-1)^{-\frac{1}{2}}$, $\lambda^{3,4,5} = 0$, and the corresponding left eigenvectors are

$$L^3 = (\tfrac{1}{2}(\gamma-1)M \;\; 1 \;\; 0 \;\; 0 \;\; 0),$$

$$L^4 = (0 \;\; 0 \;\; 0 \;\; 1 \;\; 0),$$

$$L^5 = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 1),$$

$$L^{1,2} = \left(0 \;\; \frac{2}{(\gamma-1)aM^2} \;\; \frac{\pm 1}{(M^2-1)^{\frac{1}{2}}} \;\; \frac{-1}{\gamma(\gamma-1)C_v M^2} \;\; 0\right).$$

Again $B(U_0) = 0$ and (2) and (3) give

$$\Pi_4 = \Pi_5 = 0,$$

$$\frac{2}{(\gamma-1)a_0 M_0^2}\Pi_2 = \frac{1}{(M_0^2-1)^{\frac{1}{2}}}\Pi_3,$$

$$\tfrac{1}{2}(\gamma-1)M_0\Pi_1 + \Pi_2 = 0,$$

and

$$\frac{2}{(\gamma-1)a_0 M_0^2}(\Pi_2)_{s'} + \frac{1}{(M_0^2-1)^{\frac{1}{2}}}(\Pi_3)_{s'} + \left(\frac{\rho\Phi}{\gamma C_v T}\frac{1}{aM(M^2-1)}\frac{\partial\bar{\sigma}}{\partial a}\right)_0 \Pi_2 = 0,$$

where we have transformed from (s,n)-co-ordinates to (s',ϕ)-co-ordinates with $s' = s$ and $\phi_s + \lambda^\phi \phi_n = 0$. Then

$$\Pi = \tilde{\Pi}_3 \exp(-\alpha s')\left\{\frac{-a_0 M_0}{(M_0^2-1)^{\frac{1}{2}}} \;\; \frac{\gamma-1}{2}\frac{a_0 M_0^2}{(M_0^2-1)^{\frac{1}{2}}} \;\; 1 \;\; 0 \;\; 0\right\},$$

where

$$\alpha = \frac{\gamma-1}{4\gamma}\left(\frac{\rho\Phi}{C_v T}\frac{M}{M^2-1}\frac{\partial\bar{\sigma}}{\partial a}\right)_0,$$

which is positive. If $s = 0$ corresponds to the beginning of the bend, then

$$1 + \int_0^{s_c} (\nabla_u \lambda^\phi)_0 \frac{\Pi}{\tilde{n}_\phi}\, ds' = 0.$$

This may be written as

$$1 + \int_0^{x_c} (\nabla_u \lambda^\phi) \frac{\Pi}{\tilde{n}_\phi}\, ds' = 0,$$

where λ^ϕ is now taken as $\tan(\mu+\theta)$, and where $\Pi/\tilde{n}_\phi$ is unaltered; μ is defined to be $\tan^{-1}\{(M_0^2-1)^{-\frac{1}{2}}\}$. Evaluating the integral, we obtain

$$x_c = -\frac{1}{\alpha}\log\left[1 - \frac{\alpha(M_0^2-1)^{\frac{3}{2}}}{(-\tilde{\theta}_n)\frac{1}{2}(\gamma+1)M_0^4 \tan\mu_0}\right].$$

It is easily shown that $-\tilde{\theta}_n \tan\mu_0 = \tilde{\theta}_s$, the initial curvature of the corner. Hence

$$x_c = -\frac{1}{\alpha}\log\left[1 - \frac{2}{\gamma+1}\frac{\alpha(M_0^2-1)^{\frac{3}{2}}}{\tilde{\theta}_s M_0^4}\right].$$

For completely frozen flow $\alpha = 0$, so that

$$x_c = \frac{2}{\gamma+1}\frac{(M_0^2-1)^{\frac{3}{2}}}{\tilde{\theta}_s M_0^4},$$

which is the result obtained by Johannesen (1952); if

$$\alpha = \tfrac{1}{2}(\gamma+1)\frac{\tilde{\theta}_s M_0^4}{(M_0^2-1)^{\frac{3}{2}}},$$

then x_c is infinite. This relation again defines a minimum value of the relaxation time.

4. The magnitude of τ_m

Let us now investigate the magnitude of τ_m, the value of the relaxation time below which the leading frozen characteristic suffers no breakdown. We have that

$$\frac{\partial\bar{\sigma}}{\partial a} = \frac{\partial\bar{\sigma}}{\partial T}\frac{\partial T}{\partial a} = C_{\text{vib}}\frac{2a}{\gamma R},$$

where C_{vib} is the specific heat of vibration. Therefore

$$c^* = \frac{1}{2}\frac{\gamma-1}{\gamma}\rho\Phi\frac{C_{\text{vib}}}{C_v}.$$

The value $\tau = \tau_m$ occurs when $c^* = \frac{1}{2}(\gamma+1)(-\tilde{u}_x)$, so that

$$\left(\frac{1}{\rho\Phi}\right)_m = \tau_m = \frac{1}{\gamma}\frac{\gamma-1}{\gamma+1}\left(\frac{1}{-\tilde{u}_x}\right)\frac{C_{\text{vib}}}{C_v}.$$

The maximum value of C_{vib} for a diatomic gas is R and is attained at high temperatures, so that τ_m has a maximum value

$$(\tau_m)_{\max} = \frac{1}{\gamma}\frac{(\gamma-1)^2}{\gamma+1}\left(\frac{1}{-\tilde{u}_x}\right),$$

and is less for lower temperatures. Now, for $\tau = \infty$, for a frozen gas, the value of t_c, $(t_c)_\infty$ is $[2/(\gamma+1)](1/-\tilde{u}_x)$. Therefore,

$$\frac{(\tau_m)_{\max}}{(t_c)_\infty} = \frac{(\gamma-1)^2}{2\gamma}.$$

For the two-dimensional steady case, we have

$$\frac{a_0(\tau_m)_{\max}}{(x_c)_\infty} = \frac{(\gamma-1)^2}{2\gamma}\frac{M_0}{M_0^2-1}.$$

Note that although, as $M_0 \to 1$, $\dfrac{a_0(\tau_m)_{\max}}{(x_c)_\infty}$ is unbounded, $(\tau_m)_{\max} \sim (M_0^2-1)^{\frac{1}{2}}$ and $(x_c)_\infty \sim (M_0^2-1)^{\frac{3}{2}}$, so that both tend to zero.

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